LIMITI NOTEVOLI

Forme indeterminate classiche

1) $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$ 2) $\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$

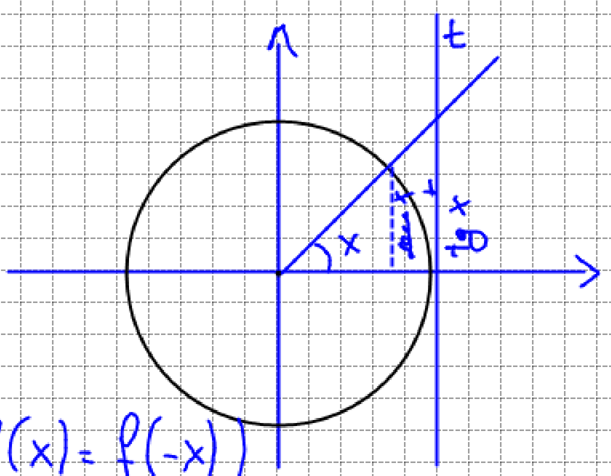
FIGLI $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$; $\lim_{x\to -\infty} \left(1+\frac{1}{x}\right)^x = e$

$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$; $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

NIPOTI $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$; $\lim_{x\to 0} \frac{\operatorname{Tg} x}{x} = 1$

$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$

Dimostrazione



1) $\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$

$f(x) = \frac{\operatorname{sen} x}{x}$ PARI $\left(f(x) = f(-x)\right)$

Quindi limitiamo il calcolo a $\lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} = 1$

$$\operatorname{sen} x \le x \le \operatorname{Tg} x \quad \forall x \in \left[0; \frac{\pi}{2}\right)$$

divido per $\operatorname{sen} x \neq 0$ $\quad 1 \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x} \quad \forall x \in \left(0; \frac{\pi}{2}\right)$

$$\cos x \le \frac{\operatorname{sen} x}{x} \le 1$$

$$\underbrace{\lim_{x\to 0^+} \cos x}_{\downarrow\ 1} \le \lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} \le \underbrace{\lim_{x\to 0^+} 1}_{\downarrow\ 1}$$

Per il teorema dei due carabinieri anche $\lim_{x\to 0^+} \frac{\operatorname{sen} x}{x} = 1$

Siccome $y = \frac{\operatorname{sen} x}{x}$ è PARI concludo che

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1$$

2) $\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$ basta assegnare ad x valori grandi e vedere a quanto tende $\left(1+\frac{1}{x}\right)^x$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \left[\frac{0}{0}\right] \text{ F.I.}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}\,\frac{(1+\cos x)}{(1+\cos x)} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} =$$

$$= \lim_{x\to 0}\frac{\operatorname{sen}^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\underbrace{\frac{\operatorname{sen} x}{x}}_{\to 1}\,\underbrace{\frac{\operatorname{sen} x}{x}}_{\to 1}\,\underbrace{\frac{1}{1+\cos x}}_{\to \frac{1}{2}} = \frac{1}{2}$$

- $$\lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x = \left[\left(1+\frac{1}{-\infty}\right)^{-\infty} \to \left(1+0^-\right)^{-\infty} \to \frac{1}{1^{\infty}}\right] \text{ F.I.}$$

$$\lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x =$$ poniamo $y=-x$ quindi quando $x\to-\infty$ $y\to+\infty$.

$$= \lim_{x\to +\infty}\left(1-\frac{1}{y}\right)^{-y} = \lim_{y\to +\infty}\left(\frac{y-1}{y}\right)^{-y} = \lim_{y\to +\infty}\left(\frac{y}{y-1}\right)^{y} =$$

$$= \lim_{y\to +\infty}\left(1+\frac{1}{y-1}\right)^{(y-1)+1} = \lim_{y\to +\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)^{1} =$$

$$a^{b+c} = a^b \cdot a^c$$

poniamo $y-1=z$ se $y\to+\infty \Rightarrow z\to+\infty$.

$$= \lim_{z\to +\infty}\underbrace{\left(1+\frac{1}{z}\right)^{z}}_{\xrightarrow{z\to+\infty} e}\underbrace{\left(1+\frac{1}{z}\right)}_{\xrightarrow{z\to+\infty} 1} = e$$

- $$\lim_{x\to 0}\frac{\ln(1+x)}{x} =$$ poniamo $\frac{1}{x} = y$

$x\to 0 \Rightarrow y\to\infty$

$$= \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\ln\left(1+\frac{1}{y}\right)^{y} =$$

$$= \ln e = 1.$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

poniamo $y=e^x-1$ $\quad x=\ln(y+1)$

$x\to 0 \Rightarrow \ln(1+y)\to 0 \Rightarrow$

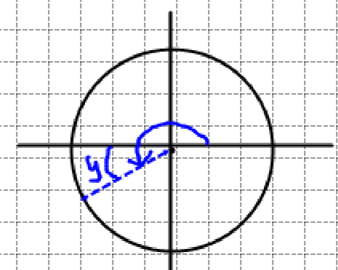
$1+y\to 1 \Rightarrow y\to 0$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{y\to 0}\frac{y}{\ln(1+y)}=\lim_{y\to 0}\frac{1}{\frac{1}{y}\ln(1+y)}=$$

$$=\lim_{y\to 0}\frac{1}{\ln\underbrace{(1+y)^{\frac{1}{y}}}_{\xrightarrow{y\to 0} e}}=1$$

stesso discorso di prima!!

- $$\lim_{x\to\pi}\frac{\operatorname{sen} x}{x-\pi}=\left[\frac{0}{0}\right] \text{ F.I.}$$



$$\lim_{x\to\pi}\frac{\operatorname{sen} x}{x-\pi}=$$ poniamo $x-\pi=y$ $\quad$ se $x\to\pi$ $y\to 0$

$$=\lim_{y\to 0}\frac{\operatorname{sen}(y+\pi)}{y}=\lim_{y\to 0}\frac{-\operatorname{sen} y}{y}=-1$$

Per casa $\displaystyle\lim_{x\to 1}\frac{\log x}{e^x-e}$